Matrix Theory over the Complex Quaternion Algebra

Yongge Tian
Department of Mathematics and Statistics
Queen's University
Kingston, Ontario, Canada K7L 3N6
email:ytian@mast.queensu.ca

Abstract. We present in this paper some fundamental tools for developing matrix analysis over the complex quaternion algebra. As applications, we consider generalized inverses, eigenvalues and eigenvectors, similarity, determinants of complex quaternion matrices, and so on.

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1. Introduction

The complex quaternion algebra (biquaternion algebra) $\mathbb{Q}$ is well known as a four dimensional vector space over the complex number field $\mathbb{C}$ with its basis 1, e_1, e_2, e_3 satisfying the multiplication laws

$$e_1^2 = e_2^2 = e_3^2 = -1, \quad e_1e_2e_3 = -1. \tag{1.1}$$

$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_3e_3 = e_2, \tag{1.2}$$

and 1 acting as unity element. In that case, any element in $\mathbb{Q}$ can be written as

$$a = a_0 + a_1e_1 + a_2e_2 + a_3e_3, \tag{1.3}$$

where a_0—$a_3 \in \mathbb{C}$. According to this definition, real numbers, complex numbers, and real quaternions all can be regarded as the special cases of complex quaternions. A well-known fundamental fact on the complex quaternion algebra $\mathbb{Q}$ (see, e. g., [4, 6, 7]) is that it is algebraically isomorphic to the 2×2 total matrix algebra $\mathbb{C}^{2\times 2}$ through the bijective map $\psi : \mathbb{Q} \longrightarrow \mathbb{C}^{2\times 2}$ satisfying

$$\psi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \psi(e_1) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \psi(e_2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \psi(e_3) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

These four matrices are well-known as Pauli matrices. Based on this map, every element $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{Q}$ has a faithful complex matrix representation as follows

$$\psi(a) := \begin{bmatrix} a_0 + a_1i & -(\, a_2 + a_3i\,) \\ a_2 - a_3i & a_0 - a_1i \end{bmatrix} \in \mathbb{C}^{2\times 2}, \tag{1.4}$$

In this article, we shall reveal a deeper relationship between a and $\psi(a)$, which can simply be stated that there is an independent invertible matrix Q of size 2 over $\mathbb{Q}$ such that all $a \in \mathbb{Q}$ satisfy the following universal similarity factorization equality

$$Q^{-1}\mathrm{diag}(\, a,\ a\,)Q = \psi(a),$$

where Q has no relation with the expression of a. Moreover we also extend this equality to all $m \times n$ matrices over $\mathbb{Q}$. On the basis of these results, we shall consider several basic problems related to complex quaternion matrices, such as, generalized inverse, eigenvalues and eigenvectors, similarity, and determinant of complex quaternion matrices.

Some known terminology on complex quaternions are listed below (see, e.g., [6]). For $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{Q}$, the *dual quaternion* of a is

$$\overline{a} = a_0 - a_1e_1 - a_2e_2 - a_3e_3; \tag{1.5}$$

the *complex conjugate* of a is

$$a^* = \overline{a_0} + \overline{a_1}e_1 + \overline{a_2}e_2 + \overline{a_3}e_3; \tag{1.6}$$

the *Hermitian conjugate* of a is

$$a^\dagger = (\overline{a})^* = \overline{a_0} - \overline{a_1}e_1 - \overline{a_2}e_2 - \overline{a_3}e_3; \tag{1.7}$$

the *weak norm* of a is

$$n(a) = a_0^2 + a_1^2 + a_2^2 + a_3^2. \tag{1.8}$$

A quaternion $a \in \mathbb{Q}$ is said to be *real* if $a^* = a$, to be *pure imaginary* if $a^* = -a$, to be *scalar* if $\overline{a} = a$, to be *Hermitian* if $a^\dagger = a$.

For any $A = (a_{st}) \in \mathbb{Q}^{m\times n}$, the *dual* of A is $\overline{A} = (\overline{a_{ts}}) \in \mathbb{Q}^{n\times m}$; the *Hermitian conjugate* of A is $A^\dagger = (a_{ts}^\dagger) \in \mathbb{Q}^{n\times m}$. A square matrix A is said to be *self-dual* if $\overline{A} = A$, it is *Hermitian* if $A^\dagger = A$, it is *unitary* if $AA^\dagger = A^\dagger A = I$, the indentity matrix, it is *invertible* if there is a matrix B over $\mathcal{Q}$ such that $AB = BA = I$.

Some known basic properties on complex quaternions and matrices of complex quaternions are listed below.

Lemma 1.1[1][6]. *Let $a,\, b \in \mathbb{Q}$ be given. Then*

(a) $\overline{\overline{a}} = a, \quad (a^*)^* = a, \quad (a^\dagger)^\dagger = a.$

(b) $\overline{a+b} = \overline{a} + \overline{b}, \qquad (a+b)^* = a^* + b^*, \qquad (a+b)^\dagger = a^\dagger + b^\dagger.$

(c) $\overline{ab} = \overline{b}\,\overline{a}, \qquad (ab)^* = a^*b^*, \qquad (ab)^\dagger = b^\dagger a^\dagger.$

(d) $a\overline{a} = \overline{a}a = n(a) = n(\overline{a});$

(e) *a is invertible if and only if $n(a) \neq 0$, in that case $a^{-1} = n^{-1}(a)\overline{a}$.*

Lemma 1.2[6]. *Let $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{n\times p}$ be given. Then*

(a) $\overline{\overline{A}} = A, \quad (A^\dagger)^\dagger = A.$

(b) $\overline{AB} = \overline{BA}, \qquad (AB)^\dagger = B^\dagger A^\dagger.$

(c) $(AB)^{-1} = B^{-1}A^{-1}$, *if A and B are invertible.*
(d) $(\overline{A})^{-1} = \overline{(A^{-1})}$, $\quad (A^{\dagger})^{-1} = (A^{-1})^{\dagger}$, *if A is invertible.*

2. A universal similarity factorization equality over complex quaternion algebra

We first present a general result on the universal similarity factorization of elements over 2×2 total matrix algebra.

Lemma 2.1. *Let $M_2(\mathbb{F})$ be the 2×2 total matrix algebra over an arbitrary field $\mathbb{F}$ with its basis e_{11}, e_{12}, e_{21} and e_{22} satisfying the following multiplication rules*

$$e_{st}e_{pq} = \begin{cases} e_{sq}, & t = p \\ 0, & t \neq p \end{cases}, \qquad s,\ t,\ p,\ q = 1,\ 2. \tag{2.1}$$

Then for any $a = a_{11}e_{11} + a_{12}e_{12} + a_{21}e_{21} + a_{22}e_{22} \in M_2(\mathcal{F})$, where $a_{st} \in \mathcal{F}$, the corresponding diagonal matrix $\mathrm{diag}(\, a,\ a\,)$ *satisfies the following universal similarity factorization equality*

$$Q \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} Q^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{F}^{2\times 2}, \tag{2.2}$$

where Q has the independent form

$$Q = Q^{-1} = \begin{bmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{bmatrix}. \tag{2.3}$$

Proof. According to Eq.(2.1), it is easy to verify that the unity element in $M_2(\mathcal{F})$ is $e = e_{11} + e_{22}$. In that case, the matrix Q in Eq.(2.3) satisfies

$$Q^2 = \begin{bmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{bmatrix} = \begin{bmatrix} e_{11}^2 + e_{21}e_{12} & e_{11}e_{21} + e_{21}e_{22} \\ e_{12}e_{11} + e_{22}e_{12} & e_{12}e_{21} + e_{22}^2 \end{bmatrix} = \begin{bmatrix} e_{11} + e_{22} & 0 \\ 0 & e_{11} + e_{22} \end{bmatrix} = eI_2,$$

which implies that Q is invertible over $M_2(\mathcal{F})$ and $Q = Q^{-1}$. Next multiplying the three matrices in the left-hand side of Eq.(2.2) yields the right-hand side of Eq.(2.2). $\square$

Theorem 2.2. *Let $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{Q}$ be given. Then the diagonal matrix* $\mathrm{diag}(\, a,\ a\,)$ *satisfies the following universal factorization similarity equality*

$$Q \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} Q = \begin{bmatrix} a_0 + a_1 i & -(\, a_2 + a_3 i\,) \\ a_2 - a_3 i & a_0 - a_1 i \end{bmatrix} = \psi(a) \in \mathbb{C}^{2\times 2}, \tag{2.4}$$

where Q is an unitary matrix over $\mathbb{Q}$

$$Q = A^{-1} = Q^{\dagger} = \frac{1}{2} \begin{bmatrix} 1 - ie_1 & e_2 + ie_3 \\ -e_2 + ie_3 & 1 + ie_1 \end{bmatrix}. \tag{2.5}$$

Proof. According to Eq.(2.4), we choose a new basis for $\mathbb{Q}$ as follows

$$e_{11} = \frac{1}{2}(\, 1 - ie_1\,), \quad e_{12} = \frac{1}{2}(\, -e_2 + ie_3\,), \quad e_{21} = \frac{1}{2}(\, e_2 + ie_3\,), \quad e_{22} = \frac{1}{2}(\, 1 + ie_2\,). \tag{2.6}$$

Then it is not difficult to verify that the above basis satisfies the multiplication rules in Eq.(2.1). Under this new basis, any element $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{Q}$ can be expressed as

$$a = (\, a_0 + ia_1 \,)e_{11} + (\, -a_2 - ia_3 \,)e_{12} + (\, a_2 - ia_3 \,)e_{21} + (\, a_0 - ia_1 \,)e_{22}. \tag{2.7}$$

Substituting Eqs.(2.6) and (2.7) into Eq.(2.2), we obtain Eqs.(2.4) and (2.5). $\square$

The equality in Eq.(2.4) can also equivalently be expressed as

$$Q \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} Q = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in \mathbb{Q}^{2\times 2}, \tag{2.8}$$

where $a_{st} \in \mathbb{C}$ is arbitrary, Q is as in Eq.(2.5), and a has the form

$$a = \frac{1}{2}(\, a_{11} + a_{22} \,) + \frac{1}{2}(\, a_{22} - a_{11} \,)ie_1 + \frac{1}{2}(\, a_{21} - a_{12} \,)e_2 + \frac{1}{2}(\, a_{12} + a_{21} \,)ie_3. \tag{2.9}$$

This equality shows that every 2×2 complex matrix is uniformly similar to an diagonal matrix with the form aI_2 over the complex quaternion algebra $\mathbb{Q}$.

The complex quaternions and their complex matrix representations satisfy the following operation properties.

Theorem 2.3 *Let $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3$, $b \in \mathbb{Q}$, $\lambda \in \mathbb{C}$ be given. Then*

(a) $a = b \Longleftrightarrow \psi(a) = \psi(b)$.

(b) $\psi(a+b) = \psi(a) + \psi(b), \quad \psi(ab) = \psi(a)\psi(b), \quad \psi(\lambda a) = \psi(a\lambda) = \lambda\psi(a), \quad \psi(1) = I_2.$

(c) $\psi(\overline{a}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \psi^T(a) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$

(d) $\psi(a^*) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overline{\psi(a)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$

(e) $\psi(a^\dagger) = \overline{\psi(a)}^T = \psi^*(a)$, *the conjugate transpose of* $\psi(a)$.

(f) $\det \psi(a) = n(a) = a_0^2 + a_1^2 + a_2^2 + a_3^2$;

(g) $a = \frac{1}{4}E_2\,\psi(a)\,E_2^\dagger$, *where* $E_2 = [\,1 - ie_1,\ e_2 + ie_3\,]$.

(h) *a is invertible if and only if $\psi(a)$ is invertible, in that case,* $\psi(a^{-1}) = \psi^{-1}(a)$ *and* $a^{-1} = \frac{1}{4}E_2\,\psi^{-1}(a)\,E_2^\dagger$.

For a noninvertible element over $\mathbb{Q}$, we can define its Moore-Penrose inverse as follows.

Definition. Let $a \in \mathbb{Q}$ be given. If the following four equations

$$axa = a, \qquad xax = x, \qquad (ax)^\dagger = ax, \qquad (xa)^\dagger = xa \tag{2.10}$$

have a common solution x, then this solution is called the Moore-Penrose inverse of a, and denoted by $x = a^+$.

The existence and the uniqueness of the Moore-Penrose inverse of a complex quaternion a can be determined by its matrix representation $\psi(a)$ over $\mathbb{C}$. In fact, according to Theorem

2.3(a), (b) and (e), the four equations in Eq.(2.10) are equivalent to the following four equations over $\mathbb{C}$

$$\psi(a)\psi(x)\psi(a) = \psi(a), \qquad \psi(x)\psi(a)\psi(x) = \psi(x),$$
$$[\,\psi(a)\psi(x)\,]^* = \psi(a)\psi(x), \qquad [\,\psi(x)\psi(a)\,]^* = \psi(x)\psi(a).$$

According to the complex matrix theory, the following four equations

$$\psi(a)Y\psi(a) = \psi(a), \qquad Y\psi(a)Y = Y, \qquad [\,\psi(a)Y\,]^* = \psi(a)Y, \qquad [\,Y\psi(a)\,]^* = Y\psi(a).$$

has a unique solution $Y = \psi^+(a)$, the Moore-Penrose inverse of $\psi(a)$. Then by Eq.(2.8), it follows that there must be a unique x over $\mathbb{Q}$ such that $\psi(x) = Y = \psi^+(a)$, in which case, this x can be expressed as

$$x = \frac{1}{4}E_2 Y E_2^\dagger = \frac{1}{4}E_2\,\psi^+(a)\,E_2^\dagger.$$

Correspondingly this x is the unique solution to Eq.(2.10). In summary, we have the following.

Theorem 2.4. *Let $a \in \mathbb{Q}$ be given. Then its Moore-Penrose inverse a^+ exists uniquely, and satisfies the following four equalities*

$$\psi(a^+) = \psi^+(a), \qquad a^+ = \frac{1}{4}E_2\,\psi^+(a)\,E_2^\dagger,$$

where $E_2 = [\,1 - ie_1,\ e_2 + ie_3\,]$.

One of the basic problems related to complex quaternions is concerned with similarity of two complex quaternions. As usual, two complex quaternions a and b are said to be similar if there is an invertible complex quaternion x such that $x^{-1}ax = b$, and this is written as $a \sim b$. It is easy to verify that the similarity mentioned here is an equivalence relation on complex quaternions.

A simple result follows immediately from the above definition, Eqs.(2.4) and (2.8).

Theorem 2.5. *Let a, $b \in \mathbb{Q}$ be given. Then*

$$a \sim b \iff \psi(a) \sim \psi(b). \tag{2.11}$$

From Eq.(2.11), we easily find the following.

Theorem 2.6. *Let $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{Q}$ given with $a \notin \mathbb{C}$.*

(a) *If $a_1^2 + a_2^2 + a_3^2 \neq 0$, then $a \sim a_0 + \tau(a)e_1$, where $\tau(a)$ is complex number satisfying $\tau^2(a) = a_1^2 + a_2^2 + a_3^2$.*

(b) *If $a_1^2 + a_2^2 + a_3^2 = 0$, then $a \sim a_0 - \frac{1}{2}e_2 + \frac{1}{2}ie_3$.*

Proof. For any $a \in \mathbb{Q}$, the characteristic polynomial of its complex matrix representation $\psi(a)$ is

$$|\,\lambda I_2 - \psi(a)\,| = \begin{vmatrix} \lambda - (\,a_0 + a_1 i\,) & a_2 + a_3 i \\ -a_2 + a_3 i & \lambda - (\,a_0 - a_1 i\,) \end{vmatrix} = (\,\lambda - a_0\,)^2 + a_1^2 + a_2^2 + a_3^2.$$

From it we immediately know that if $a_1^2 + a_2^2 + a_3^2 \neq 0$, then

$$\psi(a) \sim \begin{bmatrix} a_0 + \tau(a)i & 0 \\ 0 & a_0 - \tau(a)i \end{bmatrix} = \psi[\, a_0 + \tau(a)e_1 \,], \tag{2.12}$$

and if $a_1^2 + a_2^2 + a_3^2 = 0$, then

$$\psi(a) \sim \begin{bmatrix} a_0 & 1 \\ 0 & a_0 \end{bmatrix} = \psi\left(a_0 - \frac{1}{2}e_2 + \frac{1}{2}ie_3 \right). \tag{2.13}$$

Correspondingly applying Eq.(2.11) to Eqs.(2.12) and (2.13) may lead to Part (a) and Part (b) of this theorem. $\square$

3. Two universal factorization equalities on complex quaternion matrices

In this section, we extend the universal similarity factorization equality in (2.4) to any $m \times n$ matrix over $\mathbb{Q}$, and give some of its consequences.

Theorem 3.1. *Let $A = A_0 + A_1e_1 + A_2e_2 + A_3e_3 \in \mathbb{Q}^{m\times n}$ be given. Then A satisfies the following universal factorization equality*

$$Q_{2m} \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} Q_{2n} = \begin{bmatrix} A_0 + A_1 i & -(\, A_2 + A_3 i\,) \\ A_2 - A_3 i & A_0 - A_1 i \end{bmatrix} := \Psi(A) \in \mathbb{C}^{2m\times 2n}, \tag{3.1}$$

where Q_{2t} has the independent form

$$Q_{2t} = Q_{2t}^{-1} = Q_{2t}^{\dagger} = \frac{1}{2}\begin{bmatrix} (\,1 - ie_1\,)I_t & (\,e_2 + ie_3\,)I_t \\ (\,-e_2 + ie_3\,)I_t & (\,1 + ie_1\,)I_t \end{bmatrix}, \qquad t = m,\ n. \tag{3.2}$$

In particular, when $m = n$, Eq.(3.1)becomes a universal similarity factorization equality over $\mathbb{Q}$.

Proof. It follows directly from multiplying out the three block matrices in the left-hand side of Eq.(3.1). $\square$

The matrix $\Psi(A)$ in Eq.(3.1) is called the complex representation of A. If setting

$$A_{11} = A_0 + A_1 i, \quad A_{12} = -(\, A_2 + A_3 i\,), \quad A_{21} = A_2 - A_3 i, \quad A_{11} = A_0 - A_1 i$$

in Eq.(3.1), then it can equivalently be expressed as

$$Q_{2m} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} Q_{2n} = \begin{bmatrix} A & O \\ 0 & A \end{bmatrix} \in \mathbb{Q}^{2m\times 2n}, \tag{3.3}$$

where $A_{st} \in \mathbb{C}^{m\times n}$ are arbitrary, Q_{2t} is as in Eq.(3.2), and A has the following form

$$A = \frac{1}{2}(\, A_{11} + A_{22}\,) + \frac{1}{2}(\, A_{22} - A_{11}\,)ie_1 + \frac{1}{2}(\, A_{21} - A_{12}\,)e_2 + \frac{1}{2}(\, A_{12} + A_{21}\,)ie_3 \in \mathbb{Q}^{m\times n}. \tag{3.4}$$

which can also be stated that for any matrix $M \in \mathbb{C}^{2m\times 2n}$, there must be a unique matrix $A \in \mathbb{Q}^{m\times n}$ such that

$$\Psi(A) = M. \tag{3.5}$$

Various operation properties on complex representation of complex quaternion matrices can easily be derived from Eq.(3.1).

Theorem 3.2. *Let $A,\ B \in \mathbb{Q}^{m\times n}$, $C \in \mathbb{Q}^{n\times p}$, and $\lambda \in \mathbb{C}$ be given. Then*

(a) $A = B \Longleftrightarrow \Psi(A) = \Psi(B)$.

(b) $\Psi(A+B) = \Psi(A) + \Psi(B), \quad \Psi(AC) = \Psi(A)\Psi(C), \quad \Psi(\lambda A) = \Psi(A\lambda) = \lambda\Psi(A)$.

(c) $\Psi(A^*) = \begin{bmatrix} O & I_n \\ -I_n & O \end{bmatrix} \Psi^T(A) \begin{bmatrix} O & -I_m \\ I_m & O \end{bmatrix}$.

(d) $\Psi_1(A^\dagger) = \Psi^*(A)$, *the conjugate transpose of* $\Psi(A)$.

(e) $A = \frac{1}{4}E_{2m}\Psi(A)E_{2n}^\dagger$, *where* $E_{2t} = [\,(\,1 - ie_1\,)I_t,\ (\,e_2 + ie_3\,)I_t\,]$, $t = m,\ n$.

(f) *A is invertible if and only if $\Psi(A)$ is invertible, in that case, $\Psi(A^{-1}) = \Psi^{-1}(A)$ and $A^{-1} = \frac{1}{4}E_{2m}\Psi^{-1}(A)E_{2m}^\dagger$.*

(g) $\Psi(A)E_{2n}^\dagger E_{2n} = E_{2m}^\dagger E_{2m}\Psi(A)$.

(h) *A is Hermitian if and only if $\Psi(A)$ is Hermitian over $\mathbb{C}$.*

(i) *A is unitary if and only if $\Psi(A)$ is unitary over $\mathbb{C}$.*

Another universal factorization equality on complex quaternion matrices is established as follows.

Theorem 3.3. *Let $A = (a_{st}) \in \mathbb{Q}^{m\times n}$, and denote $D_A = (a_{st}I_2) \in \mathbb{Q}^{2m\times 2n}$. Then D_A satisfies the following universal factorization equality*

$$P_{2m}D_AP_{2n} = \begin{bmatrix} \psi(a_{11}) & \cdots & \psi(a_{1n}) \\ \vdots & & \vdots \\ \psi(a_{m1}) & \cdots & \psi(a_{mn}) \end{bmatrix} := \psi(A) \in \mathbb{C}^{2m\times 2n}, \tag{3.6}$$

where P_{2t} has the form

$$P_{2t} = P_{2t}^{-1} = P_{2t}^\dagger = \mathrm{diag}(\,Q,\ \cdots, Q\,), \quad Q = Q^{-1} = Q^\dagger = \frac{1}{2}\begin{bmatrix} 1 - ie_1 & e_2 + ie_3 \\ -e_2 + ie_3 & 1 + ie_1 \end{bmatrix}. \quad t = m,\ n.$$

In particular, when $m = n$, Eq.(3.6) becomes a universal similarity factorization equality over $\mathbb{Q}$.

Proof. Observe from Eq.(2.1) that $Q(a_{st}I_2)Q = \psi(a_{st})$. We immediately obtain

$$P_{2m}D_AP_{2n} = [\,Q(a_{st}I_2)Q\,]_{m\times n} = [\,\psi(a_{st})\,]_{m\times n} = \psi(A).$$

which is exactly Eq.(3.6). $\square$

The complex matrix $\psi(A)$ in Eq.(3.6) is also called the complex representation of A. Clearly the two complex matrices $\Psi(A)$ in Eq.(3.1) and $\psi(A)$ Eq.(3.6) are permutationally equivalent, that is, there are two permutation matrices G and H such that $G\Psi(A)H = \psi(A)$.

For convenience of application, Eq.(3.6) can be simply stated that for any $M \in \mathbb{C}^{2m\times 2n}$, there must exist an $A \in \mathbb{Q}^{m\times n}$ such that

$$\psi(A) = M. \tag{3.7}$$

Theorem 3.4. *Let $A,\, B \in \mathbb{Q}^{m\times n}$, $C \in \mathbb{Q}^{n\times p}$, $\lambda \in \mathbb{C}$ be given. Then*

(a) $A = B \Longleftrightarrow \psi(A) = \psi(B)$.

(b) $\psi(A+B) = \psi(A) + \psi(B), \quad \psi(AC) = \psi(A)\psi(C), \quad \psi(\lambda A) = \psi(A\lambda) = \lambda\psi(A)$.

(c) $\psi(I_m) = I_{2m}$.

(d) $\psi(A^{\dagger}) = \psi^{*}(A)$, *the conjugate transpose of* $\psi(A)$.

(e) *A is invertible if and only if $\psi(A)$ is invertible, in that case,* $\psi(A^{-1}) = \psi^{-1}(A)$.

(h) *A is Hermitian if and only if $\psi(A)$ is Hermitian.*

(i) *A is unitary if and only if $\psi(A)$ is unitary.*

Just as for complex quaternions, we can define the Moore-Penrose inverse of any $m \times n$ complex quaternion matrix as follows.

Definition. Let $A \in \mathbb{Q}^{m\times n}$ be given. If the following four equations

$$AXA = A, \qquad XAX = X, \qquad (AX)^{\dagger} = AX, \qquad (XA)^{\dagger} = XA \tag{3.8}$$

have a common solution for X, then this solution is called the Moore-Penrose inverse of A, and denoted by $X = A^{+}$.

The existence and the uniqueness of the Moore-Penrose inverse of a complex quaternion matrix A can be determined by its complex representation $\Psi(A)$. In fact, according to Theorem 3.2(a), (c) and (d), the four equations in Eq.(3.8) are equivalent to the following four complex matrix equations

$$\Psi(A)\Psi(X)\Psi(A) = \Psi(A), \qquad \Psi(X)\Psi(A)\Psi(X) = \Psi(X),$$

$$[\,\Psi(A)\Psi(X)\,]^{*} = \Psi(A)\Psi(X), \qquad [\,\Psi(X)\Psi(A)\,]^{*} = \Psi(X)\Psi(A).$$

According to the complex matrix theory, the following four equations

$$\Psi(A)Y\Psi(A) = \Psi(A), \qquad Y\Psi(A)Y = Y, \qquad [\,\Psi(A)Y\,]^{*} = \Psi(A)Y, \qquad [\,Y\Psi(A)\,]^{*} = Y\Psi(A)$$

have a unique common solution $Y = \Psi^{+}(A)$, the Moore-Penrose of $\Psi(A)$. Then by Eq.(3.5), there is a unique matrix X over $\mathbb{Q}$ such that $\Psi(X) = Y = \Psi^{+}(A)$, in which case, this X can be expressed as

$$X = \frac{1}{4}E_{2n}YE_{2m}^{\dagger} = \frac{1}{4}E_{2n}\Psi^{+}(A)E_{2m}^{\dagger}.$$

Correspondingly this X is the unique solution to Eq.(3.8). Hence we have the following.

Theorem 3.5. *Let $A \in \mathbb{Q}^{m\times n}$ be given. Then its Moore-Penrose inverse A^{+} of A exists uniquely, and satisfies the following two equalities*

$$\Psi(A^{+}) = \Psi^{+}(A), \qquad A^{+} = \frac{1}{4}E_{2n}\Psi^{+}(A)E_{2m}^{\dagger},$$

where $E_{2t} = [\,(1 - ie_1\,)I_t,\ (\,e_2 + ie_3\,)I_t\,],\ t = m,\ n.$

Through the complex representation in Eq.(3.1), we can define the rank of any $A \in \mathbb{Q}^{m\times n}$ as follows $\mathrm{rank}(A) = \frac{1}{2}\mathrm{rank}[\,\Psi(A)\,]$. Obviously the rank of a complex quaternion matrix A is a fraction if the rank of $\Psi(A)$ is an odd number. In particular, $A \in \mathbb{Q}^{m\times m}$ is invertible if and only if $\mathrm{rank}(A) = m$.

On the basis of the above results we now are able to investigate various kinds of problems related to complex quaternion matrices. In the next three sections, we consider three basic problems—right eigenvalues and eigenvectors, similarity, as well as determinants of square complex quaternion matrices.

4. Right eigenvalues and eigenvectors of complex quaternion matrices

As usual, *right eigenvalue equation* for a complex quaternion matrix $A \in \mathbb{Q}^{n\times n}$ is defined by

$$AX = X\lambda, \qquad X \in \mathbb{Q}^{n\times 1}, \qquad \lambda \in \mathbb{Q}. \tag{4.1}$$

If a $\lambda \in \mathbb{Q}$ and a nonzero $X \in \mathbb{Q}^{n\times 1}$ satisfy Eq.(4.1), then λ is called a *right eigenvalue* of A and X is called an *eigenvector* associated with λ. In particular, if a $\lambda \in \mathbb{Q}$ and an $X \in \mathbb{Q}^{n\times 1}$ with $\mathrm{rank}(X) = 1$ satisfy Eq.(4.1), then λ is called a *regular right eigenvalue* of A and X is called an *regular eigenvector* associated with λ. In this section, we shall prove that any square complex quaternion matrix has at least one complex right eigenvalue, and also has at least one regular right eigenvalue. To do so, we need some preparation.

Definition. For any $X = X_0 + X_1e_1 + X_2e_2 + X_3e_3 \in \mathbb{Q}^{n\times 1}$, we call the following complex matrix

$$\begin{bmatrix} X_0 + X_1 i \\ X_2 - X_3 i \end{bmatrix} := \overrightarrow{X} \in \mathbb{C}^{2n\times 1}, \tag{4.2}$$

uniquely determined by X, the *complex adjoint vector* of X.

Lemma 4.1. *Let* $A \in \mathbb{Q}^{m\times n}$, $X \in \mathbb{Q}^{n\times 1}$, *and* $\lambda \in \mathbb{C}$ *be given. Then* $\overrightarrow{AX} = \Psi(A)\overrightarrow{X}$ *and* $\overrightarrow{X\lambda} = \overrightarrow{X}\lambda$.

Proof. It is easy to see that for any column matrix $Y \in \mathbb{Q}^{n\times 1}$, the corresponding $\Psi(Y)$ and $\overrightarrow{Y}$ satisfy the relation $\overrightarrow{Y} = \Psi(Y)[\,1,\ 0\,]^T$. Now applying it to AX and $X\lambda$ we find

$$\overrightarrow{AX} = \Psi(AX)[\,1,\ 0\,]^T = \Psi(A)\Psi(X)[\,1,\ 0\,]^T = \Psi(A)\overrightarrow{X},$$
$$\overrightarrow{X\lambda} = \Psi(X\lambda)[\,1,\ 0\,]^T = \Psi(X)\lambda[\,1,\ 0\,]^T = \overrightarrow{X}\lambda. \qquad \square$$

Based on the above notation, we can deduce the following several results.

Theorem 4.2. *Let* $A \in \mathbb{Q}^{n\times n}$ *be given. Then all the eigenvalues of* $\Psi(A)$ *are right eigenvalues of* A, *and all the eigenvectors of* $\Psi(A)$ *can be used for constructing eigenvectors of* A.

Proof. Assume that $\lambda \in \mathbb{C}$ and $O \neq Y \in \mathbb{C}^{2n\times 1}$ satisfy

$$\Psi(A)Y = Y\lambda.$$

Then we set $X = E_{2n}Y \in \mathbb{Q}^{n\times 1}$, where $E_{2n} = [\,(1 - ie_1)I_n,\ (e_2 + ie_3)I_n\,]$, and it satisfies $E_{2n}E_{2n}^{\dagger} = 4I_n$. Combining the above results with Theorem 3.2(e) and (g), we obtain

$$AX = \frac{1}{4}E_{2n}\Psi(A)E_{2n}^{\dagger}E_{2n}Y = \frac{1}{4}E_{2n}E_{2n}^{\dagger}E_{2n}\Psi(A)Y = E_{2n}Y\lambda = X\lambda,$$

which shows that λ is a right eigenvalue of A and $X = E_{2n}Y$ is an eigenvector of A associated with this λ. So the conclusion of the theorem is true. $\square$

Conversely, we have the following result.

Theorem 4.3. *Let $A \in \mathbb{Q}^{n\times n}$ be given. Then all the complex right eigenvalues of A are eigenvalues of $\Psi(A)$, too.*

Proof. Assume that $\lambda \in \mathbb{C}$ and $O \neq X \in \mathbb{Q}^{n\times 1}$ satisfy $AX = X\lambda$. Then applying Lemma 4.1 to the both sides of this equality, we find $\Psi(A)\overrightarrow{X} = \overrightarrow{X}\lambda$, which shows that λ and $\overrightarrow{X}$ are a pair of eigenvalue and eigenvector of $\Psi(A)$. $\square$

Next are two results on the regular right eigenvalues and eigenvectors of complex quaternion matrices.

Theorem 4.4. *Let $A \in \mathbb{Q}^{n\times n}$ be given. Then A has at least one regular right eigenvalue.*

Proof. We first assume that the complex representation $\Psi(A)$ has at least two linearly independent eigenvectors Y_1 and Y_2, i.e.,

$$\Psi(A)Y_1 = Y_1\lambda_1, \qquad \Psi(A)Y_2 = Y_2\lambda_2, \tag{4.3}$$

where $\mathrm{rank}[\,Y_1,\ Y_2\,] = 2$, λ_1 and λ_2 are two complex numbers with $\lambda_1 = \lambda_2$, or $\lambda_1 \neq \lambda_2$. Then

$$\Psi(A)[\,Y_1,\ Y_2\,] = [\,Y_1,\ Y_2\,]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{4.4}$$

According to Eq.(3.5), there must exist an $X \in \mathbb{Q}^{n\times 1}$ such that

$$\Psi(X) = [\,Y_1,\ Y_2\,], \qquad \mathrm{rank}(X) = \frac{1}{2}\mathrm{rank}\Psi(X) = 1, \tag{4.5}$$

meanwhile there must exist a $\lambda \in \mathbb{Q}$ such that

$$\Psi(\lambda) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{4.6}$$

Putting Eqs.(4.5) and (4.6) into Eq.(4.4) and then applying Theorem 3.2(a) to it, we obtain

$$\Psi(A)\Psi(X) = \Psi(X)\Psi(\lambda) \Longrightarrow AX = X\lambda, \ \ \textit{and} \ \ \mathrm{rank}(X) = 1,$$

which shows that λ and X are a pair of regular right eigenvalue and eigenvector of A.

Next assume that the complex matrix $\Psi(A)$ has only one eigenvalue λ_1 and only one linearly independent eigenvector Y_1 associated with λ_1. Then according to complex matrix theory, there exists another vector Y_2 over $\mathbb{C}$ such that

$$\Psi(A)[\,Y_1,\ Y_2\,] = [\,Y_1,\ Y_2\,]\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}, \tag{4.7}$$

where $\mathrm{rank}[\,Y_1,\ Y_2\,] = 2$. Then it follows by Eq.(3.5) that there must exist a $\lambda \in \mathbb{Q}$ and an $X \in \mathbb{Q}^{n\times 1}$ such that

$$\Psi(\lambda) = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}, \quad \Psi(X) = [\,Y_1,\ Y_2\,], \quad \mathrm{rank}(X) = \frac{1}{2}\mathrm{rank}\Psi(X) = 1. \tag{4.8}$$

In that case, combining Eqs.(4.7) and (4.8) and Theorem 3.2(a), we get

$$\Psi(A)\Psi(X) = \Psi(X)\Psi(\lambda) \implies AX = X\lambda, \quad r(X) = 1,$$

which shows that λ and X are a pair of regular right eigenvalue and eigenvector of A. $\square$

Theorem 4.5. *Let $A \in \mathbb{Q}^{n\times n}$ be given. Then from any regular right eigenvalue of A we can derive eigenvalues of $\Psi(A)$.*

Proof. Assume that $\lambda = \lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in \mathbb{Q}$ and $X \in \mathbb{Q}^{n\times 1}$ with $\mathrm{rank}(X) = 1$ satisfy

$$AX = X\lambda. \tag{4.9}$$

If $\lambda \in \mathbb{C}$ in Eq.(4.9), then from Theorem 3.3(a) and (b) we get

$$\Psi(A)\Psi(X) = \Psi(X)\Psi(\lambda) = \Psi(X)\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix},$$

which clearly shows that λ is an eigenvalue of $\Psi(A)$, and $\Psi(X)$ are two eigenvectors of $\Psi(A)$. If $\lambda \notin \mathbb{C}$ in Eq.(4.9), then by Theorem 2.6 we know that this λ can be expressed as

$$\lambda = p[\,\lambda_0 + \tau(\lambda)e_1\,]p^{-1}, \tag{4.10}$$

when $\tau^2(\lambda) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \neq 0$, or

$$\lambda = q\left(\lambda_0 - \frac{1}{2}e_2 + \frac{1}{2}ie_3\right)q^{-1}, \tag{4.11}$$

when $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$. Under the condition in (4.10), the equality in Eq.(4.9) can be equivalently expressed as

$$A(Xp) = (Xp)[\,\lambda_0 + \tau(\lambda)e_1\,]. \tag{4.12}$$

Applying Theorem 3.2(b) to the both sides of Eq.(4.12) yields

$$\Psi(A)\Psi(Xp) = \Psi_1(Xp)\Psi[\,\lambda_0 + \tau(\lambda)e_1\,] = \Psi(Xp)\begin{bmatrix} \lambda_0 + \tau(\lambda)i & 0 \\ 0 & \lambda_0 - \tau(\lambda)i \end{bmatrix},$$

which shows that $\lambda_0 \pm \tau(\lambda)i$ are two eigenvalues of $\Psi(A)$, and $\Psi(Xp)$ are two eigenvectors of $\Psi(A)$. Under the condition in Eq.(4.11), the equality in (4.9) can equivalently be expressed as

$$A(Xq) = (Xq)\left(\lambda_0 - \frac{1}{2}e_2 + \frac{1}{2}ie_3\right). \tag{4.13}$$

Then applying Theorem 3.2(b) to the both sides of Eq.(4.13) yields

$$\Psi(A)\Psi(Xq) = \Psi(Xq)\Psi\left(\lambda_0 - \frac{1}{2}e_2 + \frac{1}{2}ie_3\right) = \Psi(Xq)\begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix},$$

which shows that λ_0 is an eigenvalue of $\Psi(A)$, and the first column of $\Psi(Xp)$ is an eigenvector of $\Psi(A)$. $\square$

5. Similarity of complex quaternion matrices

Two square matrices A and B of the same size over $\mathbb{Q}$ are said to be *similar* if there exists an invertible matrix X over $\mathbb{Q}$ such that $X^{-1}AX = B$. Based on the results in the preceding two subsections, we can easily find a simple result charactering the similarity of two complex quaternion matrices.

Theorem 5.1. *Let $A,\, B \in \mathbb{Q}^{n\times n}$ be given. Then the following three statements are equivalent:*

(a) *A and B are similar over $\mathbb{Q}$.*

(b) *$\Psi(A)$ and $\Psi(B)$ are similar over $\mathbb{C}$.*

(c) *$\psi(A)$ and $\psi(B)$ are similar over $\mathbb{C}$.*

Proof. Assume first that $A \sim B$ over $\mathbb{Q}$. Then there is an invertible matrix X such that $X^{-1}AX = B$. Now applying Theorem 3.2(b) and (f) to its both sides yields $\Psi^{-1}(X)\Psi(A)\Psi(X) = \Psi(B)$, which shows that $\Psi(A) \sim \Psi(B)$ over $\mathbb{C}$. Conversely, assume that $\Psi(A) \sim \Psi(B)$ over $\mathbb{C}$. Then there is an invertible matrix $Y \in \mathbb{C}^{2n\times 2n}$ such that $Y^{-1}\Psi(A)Y = \Psi(B)$. For this Y, by (3.5) there must be an invertible matrix $X \in \mathbb{Q}^{n\times n}$ such that $\Psi(X) = Y$. Thus, $\Psi(X^{-1})\Psi(A)\Psi(X) = \Psi(B)$, and consequently $X^{-1}AX = B$, which shows that $A \sim B$. The equivalence of (a) and (c) can also be shown in the same manner. $\square$

Based on this result, we can extend various results on similarity of complex matrices to complex quaternion matrices.

Theorem 5.2. *Let $A \in \mathbb{Q}^{n\times n}$ be given. Then A is similar to a diagonal matrix over $\mathbb{Q}$ if and only if the sizes of the Jordan blocks in the Jordan canonical form of the complex representation $\psi(A)$ of A are not greater than 2.*

Proof. Assume that A is diagonalizable over $\mathbb{Q}$, i.e., there is an invertible matrix P over $\mathbb{Q}$ such that

$$P^{-1}AP = \mathrm{diag}(\, \lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n\,), \tag{5.1}$$

where λ_1—$\lambda_n \in \mathbb{Q}$. Then by Theorem 5.1, we obtain

$$\psi^{-1}(P)\psi(A)\psi(P) = \mathrm{diag}(\, \psi(\lambda_1),\ \psi(\lambda_2),\ \cdots,\ \psi(\lambda_n)\,), \tag{5.2}$$

where $\psi(\lambda_t)$ is a 2×2 complex matrix. This equality implies that the sizes of Jordan block in the Jordan canonical form of $\psi(A)$ are not greater than 2. Conversely assume that there is an invertible matrix G over $\mathbb{C}$ such that

$$G^{-1}\psi(A)G = \mathrm{diag}(\, J(\mu_1),\ \cdots,\ J(\mu_k),\ \mu_{k+1},\ \cdots,\ \mu_r\,), \tag{5.3}$$

where $J(\mu_t)(t = 1$—$k)$ is a 2×2 complex Jordan block, μ_{k+1}—μ_r are complex eigenvalues of $\psi(A)$ with $2k + (r-k) = 2n$. In that case, by Eq.(3.7) we know that there is an invertible

matrix P over $\mathbb{Q}$ such that $\phi(P) = G$, there is a λ_t over $\mathbb{Q}$ such that $\psi(\lambda_t) = J(\mu_t)(t = 1—k)$, and there are $\lambda_{k+1}, \cdots, \lambda_r \in \mathbb{Q}$ such that

$$\psi(\lambda_{k+1}) = \begin{bmatrix} \mu_{k+1} & 0 \\ 0 & \mu_{k+2} \end{bmatrix}, \ \psi(\lambda_{k+2}) = \begin{bmatrix} \mu_{k+3} & 0 \\ 0 & \mu_{k+4} \end{bmatrix}, \ \cdots, \ \psi(\lambda_n) = \begin{bmatrix} \mu_{r-1} & 0 \\ 0 & \mu_r \end{bmatrix}, \tag{5.4}$$

In that case, let $D = \mathrm{diag}(\lambda_1, \ \lambda_2, \ \cdots, \ \lambda_n)$. Then Eq.(5.3) becomes

$$\psi^{-1}(P)\psi(A)\psi(P) = \psi(D) \implies P^{-1}AP = D.$$

Hence A is diagonalizable over $\mathbb{Q}$. $\square$.

The result in the above theorem alternatively implies that if the size of a Jordan block in the Jordan canonical form of the complex representation $\psi_(A)$ of A is greater than 2, then A can not be diagonalizable over $\mathbb{Q}$.

Theorem 5.3. *Let $A \in \mathbb{Q}^{n\times n}$ be given. Then A is similar to a complex matrix over $\mathbb{Q}$ if and only if the complex representation $\Psi(A)$ of A is similar to a block diagonal matrix* $\mathrm{diag}(J, \ J)$ *over $\mathbb{C}$, where $J \in \mathbb{C}^{n\times n}$.*

Proof. If $A \sim J \in \mathbb{C}^{n\times n}$, then it follows by Theorem 5.1 and Eq.(3.1) that $\Psi(A) \sim \Psi(J) = \mathrm{diag}(J, \ J)$. Conversely if $\Psi(A) \sim \mathrm{diag}(J, \ J)$ with $J \in \mathbb{C}^{n\times n}$, then it follows by (3.1) that $\Psi(J) = \mathrm{diag}(J, \ J)$. Consequently $A \sim J$ follows by Theorem 5.1. $\square$

6. Determinants of complex quaternion matrices

As one the most fundamental problems in the theory of quaternion matrices, the determinants of complex quaternion matrices, including the determinants of real quaternion matrices, have been considered by lots of authors from different aspects, see, e.g., [3], [5], [6] and [10]. Among all of them, a direct and simple method for defining determinants of quaternion matrices is through their representations in the central fields of the corresponding quaternion algebras. In [10], Zhang presents this kind of definition for determinants of real quaternion matrices, and demonstrates the consistency of his definition with some other classic definitions on determinants based on the products and sums of entries in matrices.

Following the introduction of the universal similarity equalities for generalized quaternion matrices, we shall easily find that it is a most reasonable method to define determinants of quaternion matrices through their representations in the central field of corresponding quaternion algebra. In fact, from Eq.(3.1), we know that for any $A \in \mathbb{Q}^{n\times n}$

$$Q_{2n}\mathrm{diag}(A, \ A)Q_{2n}^{-1} = \Psi(A) \in \mathbb{C}^{2n\times 2n}. \tag{6.1}$$

Therefore if we hope that determinants of quaternion matrices over $\mathbb{Q}$ satisfy the following two basic properties

$$\det(MN) = \det M \det N \quad \textit{and} \quad \det I_n = 1$$

for any $M, \ N \in \mathbb{Q}^{n\times n}$, then as a natural consequence of these two properties, the determinant of the diagonal matrix $\mathrm{diag}(A, \ A)$ satisfies the following equality

$$\det[\,\mathrm{diag}(A, \ A)\,] = \det[\,Q_{2n}\mathrm{diag}(A, \ A)Q_{2n}^{-1}\,] = \det\Psi(A). \tag{6.2}$$

From this equality we see that there seem only two kinds of natural choices for the definition of determinant of $A \in \mathbb{Q}^{n\times n}$, namely, define

$$\det A := |\Psi(A)|, \tag{6.3}$$

or

$$\det A := |\Psi(A)|^{\frac{1}{2}}, \tag{6.4}$$

where $|\Psi(A)|$ is the ordinary determinant of the complex matrix $\Psi(A)$. For simplicity, we prefer Eq.(6.3) as the definition of determinant of matrix over $\mathbb{Q}$ and call it the *central determinant* of matrix A, and denote it by $|A|_c$.

Some basic operation properties on the central determinants of complex quaternion matrices are listed below without proofs.

Theorem 6.1. *Let A, $B \in \mathbb{Q}^{n\times n}$, $\lambda \in \mathbb{C}$, and $\mu \in \mathbb{Q}$ be given.*

(a) *If $A \in \mathbb{C}^{n\times n}$, then $|A|_c = |A|^2$.*

(b) *A is invertible $\Longleftrightarrow |A|_c \neq 0$.*

(c) $|AB|_c = |A|_c\,|B|_c$.

(d) $|\lambda A|_c = |A\lambda|_c = \lambda^{2n}|A|_c$.

(e) $|\mu A|_c = |A\mu|_c = n^{2n}(\mu)|A|_c$, *where $n(\mu)$ is the weak norm of μ.*

(f) $|A^{-1}|_c = |A|_c^{-1}$.

(g) $|A^{\dagger}|_c = \overline{|A|}_c$.

(h) *If* $A = \begin{bmatrix} a_{11} & \cdots & * \\ & \ddots & \vdots \\ & & a_{nn} \end{bmatrix}$, *then* $|A|_c = n(a_{11})n(a_{22})\cdots n(a_{nn})$.

(i) *If* $A = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}$, *where A_1 and A_2 are square, then $|A|_c = |A_1|_c|A_2|_c$.*

(j) *If $A \sim B$, then $|A|_c = |B|_c$.*

By means of the determinants of complex quaternion matrices defined above, we can easily define the central characteristic polynomial of any $A \in \mathbb{Q}^{n\times n}$ as follows

$$p_A(\lambda) = |\lambda I_{2n} - \Psi(A)|, \tag{6.5}$$

which is a complex polynomial of degree $2n$. From it we easily get the following.

Theorem 6.2(Cayley-Hamilton theorem). *Let $A \in \mathbb{Q}^{n\times n}$ be given. Then $p_A(A) = 0$.*

Proof. Since $p_A(\lambda) = |\lambda I_{2n} - \Psi(A)|$, so $p_A[\Psi(A)] = 0$. Putting Eq.(6.1) in it and simplifying the equality yields

$$p_A[\,\mathrm{diag}(A,\ A)\,] = \mathrm{diag}(\,p_A(A),\ p_A(A)\,) = 0,$$

which implies that $p_A(A) = 0$. $\square$

7. Conclusions

In the article, we have established a fundamental universal similarity factorization equality over the complex quaternion algebra $\mathbb{Q}$. This equality clearly reveals the intrinsic relationship between the complex quaternion algebra $\mathbb{Q}$ and the 2×2 total matrix algebra, and could serve as a powerful tool for investigating various problems related to complex quaternions and their applications. In addition to the results in Sections 3—6, we can also apply Eqs.(2.4), (3.1) and (3.6) to investigate some other basic topics related to complex quaternion matrices, such as, singular value decompositions, norms, numerical ranges of complex quaternion matrices and so on. Finally we point out that the equality (2.4) can also extended to the complex Clifford algebra $\mathbb{C}_n$, and a set of perfect matrix representation theory on the complex Clifford algebra $\mathbb{C}_n$ can explicitly established. We shall examine this topic in another paper.

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